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LETTER TO THE EDITOR

Low temperature behaviour of the random field Ising model

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Abstract. A relationship is established between the Ising model in a random positive and negative magnetic field on a lattice with $p=\frac{1}{2}$. At $T=0$ each percolation cluster gives rise to a first-order transition at a different value of the field. There is a significant difference in behaviour between lattices for which $p_c<\frac{1}{2}$ which have an infinite cluster, and $p_c>\frac{1}{2}$ which do not. By considering the Bethe lattice and allowing the coordination number to become large, the results of the mean field approximation are reproduced. The above considerations do not apply to a Gaussian distribution of fields, and the absence of a first-order transition can be understood in this case. Since large clusters overturn for small fields there are clear indications of metastable behaviour.

Evidence has been accumulating that the critical behaviour of the Ising model in a random field is much more complex than had been thought at first. There were strong indications that for dimension $d>4$ the critical behaviour of the random model parallels that of the standard model in $(d-2)$ dimensions (Aharony *et al* 1976, Young 1977, Parisi and Sourlas 1979). However, this may well be an oversimplification (Schwartz 1983). Also the question of the lower critical dimension for which there is no transition is still unresolved, alternative arguments having been advanced to support $d_l=2$ (e.g. Grinstein and Ma 1982) and $d_l=3$ (e.g. Pytte *et al* 1981, Niemi 1982). (For a general review of the current situation see Imry (1983).)

Even in mean field treatment, the model shows a surprising dependence on the form of the random field distribution. Calculations for a model with a Gaussian distribution of fields by Schneider and Pytte (1977) led to a line of second-order transitions separating the ferromagnetic and 'spin glass' phases. But for a δ-function distribution with equal probabilities of positive and negative fixed fields, $\pm H_0$, Aharony (1978) showed that there is a tricritical point, and that at sufficiently low temperatures the phase transition becomes first order. Precise conditions on the probability distribution for obtaining a second- or first-order transition at $T=0$ in the mean field treatment have been discussed by Andelman (1983), and are quite complex; it is not sufficient to look at the sign of the second coefficient in a Landau expansion of the free energy.

Our own investigations have been concerned with the derivation of linked-cluster expansions of 'excitation' type (Domb 1970) to describe the disordering of the ferromagnetic state. In the course of these investigations we studied the behaviour of short-range force lattice models with a δ-function distribution of random fields as a function of the magnitude of the field at $T=0$, and we were surprised to find that the

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system undergoes a hierarchy of first-order phase transitions in a manner reminiscent of the Griffiths singularities for a dilute ferromagnet (Griffiths 1969, Domb 1974a). This applies to lattice models in all dimensions. Similar results for the SQ lattice in two dimensions were reported by Morgenstern *et al* (1981), and non-analytic behaviour has been reported recently by Schwartz *et al* (1983).

We have established a correlation between the behaviour of the model and site percolation on the lattice with probability $p=\frac{1}{2}$. A significant feature of the problem is the value of the critical percolation probability p_c for the lattice. If $p_c<\frac{1}{2}$ there is an infinite cluster at $p=\frac{1}{2}$ which gives rise to the dominant first-order transition at $T=0$.

We looked at the model for a Bethe lattice in order to see how its behaviour differs from that of the standard lattice models. In the limiting case of large coordination we were led to the mean field treatment.

Finally, we studied the corresponding behaviour of the Gaussian distribution and were able to see qualitatively how the first-order phase transitions disappear.

For simplicity we deal with the spin-$\frac{1}{2}$ Ising model with Hamiltonian

$$\mathcal{H}=-J\sum_{\langle i,j\rangle}\sigma_i\sigma_j-m\sum_i H_i\sigma_i \tag{1}$$

using standard notation (Domb 1974b). The H_i are distributed randomly with probability $p(H)\,\mathrm{d}H$ which here we shall take to be

$$p_\delta(H)=\tfrac{1}{2}[\delta(H-H_0)+\delta(H+H_0)]. \tag{2}$$

For sufficiently small H_0 ($T=0$) the coupling between neighbours dominates, and the lowest energy state is ferromagnetic with energy

$$E_{0\mathrm{f}}=-\tfrac{1}{2}qJ. \tag{3}$$

However, when H_0 becomes sufficiently large the coupling between neighbours is broken and each spin orients in the direction of its own magnetic field. Following Schneider and Pytte (1977) we shall term this the spin-glass phase, since the spin-glass order parameter

$$Q=\langle\langle\sigma_i\rangle^2\rangle_h \tag{4}$$

is equal to 1 for this phase. The energy of the spin-glass phase

$$E_{0\mathrm{sg}}=-mH_0. \tag{5}$$

From energetic considerations above a transition from ferromagnetic to spin-glass phase should take place at a value of H_0 given by

$$mH_0=\tfrac{1}{2}qJ. \tag{6}$$

However, this is an 'average' consideration. When we come to look at the conditions for the overturn of individual spins we will encounter much greater complexity.

Let us first analyse the pattern of local fields in the ferromagnetic phase. One half of the spins on average will be oriented parallel to their local fields. The other half which are oriented antiparallel to their local fields can be put into correspondence with a site percolation process on the lattice with concentration $p=\frac{1}{2}$. In this process a fraction α_1 of sites will consist of isolated singlets, α_2 of doublets, $\alpha_3^{(1)}$, $\alpha_3^{(2)}$ and $\alpha_3^{(3)}$ of the various types of triplet cluster, and so on; the fractions $\alpha_i^{(j)}$ can readily be estimated from the perimeter polynomials for the lattice (see e.g. Domb 1983). If there is an infinite cluster present we must add a term α_∞ to account for the fraction of sites contained in the 'infinite' cluster, and we have the relation

$$\sum_{i,j} \alpha_i^{(j)} = \tfrac{1}{2}. \tag{7}$$

For most two-dimensional lattices there will be no infinite cluster at $p=\frac{1}{2}$, but for lattices in three or more dimensions there will usually be an infinite cluster.

Consider now the field needed to overturn a singlet spin. The coupling energy lost in such an overturn will be $2qJ$, and hence such an overturn will not occur until the field H_0 reaches a value given by

$$mH_{01} = qJ. \tag{8}$$

The corresponding coupling energy lost in the overturn of a doublet will be $4(q-1)J$, the gain in magnetic energy will be $4mH_0$, and the overturn will take place at a lower field

$$mH_{02} = (q-1)J. \tag{9}$$

We can proceed similarly to find the value of H_0 at which any r-cluster will overturn. For compact clusters the value of H_0 will decrease steadily with increasing r, since the gain in magnetic energy will be given by $2rmH_0$ and the loss of coupling energy will be of order $Jr^{d-1/d}$. However small the field, it will be possible to find a sufficiently large cluster for which an overturn is energetically favoured. But the fraction of spins involved in such an overturn will be very small.

For ramified clusters H_0 will reach a non-zero limit as r increases, and for the infinite cluster (which is ramified) this limit is given by

$$mH_{0\infty} = \tfrac{1}{2}qJ \tag{10}$$

for all lattices since the distribution of neighbours in the infinite cluster is random (see e.g. Domb 1983).

Let us now consider the behaviour of the model as H_0 increases. For each critical field $H_{0r}^{(j)}$ corresponding to the overturn of a particular r-cluster there will be a first-order phase transition, the ferromagnetic order parameter

$$R = \langle \sigma_i \rangle \tag{11}$$

dropping discontinuously by an amount $\alpha_r^{(j)}$. For small H_0 these critical fields will be densely packed, the point $H_0 = 0$ being a point of accumulation, and the amounts $\alpha_r^{(j)}$ will be small. But as H_0 increases it will be possible to distinguish individual critical values and discontinuities, until at a critical field $2qJ/m$ the final jump of α_1 will occur to $R=0$.

The above pattern of behaviour was described for the SQ lattice by Morgenstern *et al* (1981) for which $q=4$ and $p_c > \frac{1}{2}$, and there is no infinite cluster. But when $p_c < \frac{1}{2}$ and there is an infinite cluster present, most of the spins will have aggregated to the infinite cluster which will provide the dominant singularity at the value of field (10) corresponding to the 'average' transition (6).

The above argument can readily be extended to the Bethe lattice. Equations (8) and (9) giving the critical field for the overturn of singlets and doublets remain valid, but all clusters are now ramified, and there is a limiting field below which no clusters can overturn. This field corresponds to large ramified clusters with minimal end effects, and is given by

$$mH_0 = (q-2)J. \tag{12}$$

The value of H_0 corresponding to (12) is now a point of accumulation of first-order transitions.

To proceed to the mean field limit we must allow J to become small and q to become large, qJ remaining finite. We then find that the contribution of finite clusters becomes negligible as $q \to \infty$, the total contribution coming from the infinite cluster for which $mH = \frac{1}{2}qJ$. At this value R drops from 1 to 0. This is the behaviour found by Aharony (1978).

We now turn to the case when the random fields follow the Gaussian distribution,

$$p(H) = [1/\sigma(2\pi)^{1/2}] \exp(-H^2/2\sigma^2). \tag{13}$$

An immediate difference from the δ-function distribution is the form of the lowest energy state. For any finite value of σ there will be overturned singlets corresponding to local fields in the tail of the distribution for which

$$mH_i > qJ. \tag{14}$$

There will be overturned doublets for pairs of neighbours satisfying

$$m(H_i + H_j) > 2(q-1)J, \tag{15}$$

and similarly for higher-order clusters.

As σ increases there are no critical values at which particular clusters are suddenly able to overturn. The fraction of r-clusters overturned changes smoothly as a function of σ, and the mechanism for generating discontinuities and first-order phase transitions has been removed.

But it is not clear whether there is any finite value of σ at which R becomes zero, giving rise to a second-order transition. For any σ, however large, there will be spins with small local fields which will not satisfy equations like (14), and it seems as if R will not become zero until σ becomes infinite. The influence of the infinite cluster and the mean field second-order transition found by Schneider and Pytte (1977) and Aharony (1978) require further investigation.

When we consider the behaviour as a function of magnetic field for $T > 0$, finite clusters no longer give rise to discontinuities but to smooth 'bumps'. The only discontinuous transition which remains corresponds to the infinite cluster and takes place at a magnetic field close to (12).

It is clear therefore that any lattice for which the critical site percolation probability $p_c \geq \frac{1}{2}$ cannot have a discontinuous transition. This applies to standard two-dimensional lattices, and is in agreement with Morgenstern *et al* (1981) for the SQ lattice. However, it is possible to find highly coordinated lattices (e.g. the SQ lattice with nearest, second and third neighbour interactions) for which $p_c < \frac{1}{2}$. If current ideas about the absence of a transition for the random field model in two dimensions are correct (see e.g. Aharony and Pytte 1983) the infinite cluster should then be unstable against small temperature perturbations. But this is worthy of direct investigation by series and Monte Carlo methods.

Standard three-dimensional lattices have $p_c < \frac{1}{2}$, and therefore give rise to an infinite cluster at $p = \frac{1}{2}$. However, it may be possible to find loosely packed lattices for which $p_c > \frac{1}{2}$ (e.g. the hydrogen peroxide lattice for which $q = 3$) for which there is no discontinuous transition. If there is always a transition in dimension $d \geq 4$, then no lattice can exist with a sufficiently loose packing to have $p_c > \frac{1}{2}$. The question of maximum looseness of packing is of considerable geometrical interest (see Hilbert and Cohn-Vossen 1952), and is clearly relevant here.

We have established a complex pattern of behaviour at the absolute zero for all short-range force lattice models. It is clearly an oversimplification to state that the random field model in d dimensions parallels the non-random field model in $d-2$ dimensions. The discontinuous transition corresponds to the behaviour of an infinite site percolation cluster in the lattice for $p=\frac{1}{2}$. But we can now understand the irregular behaviour of high and low temperature series expansions, since there are finite cluster effects on both sides of the transition. It might be advisable to develop series expansions for the infinite cluster alone.

Several of the effects which have manifested themselves are non-universal, e.g. the dependence of critical behaviour on the form of the probability distribution $p(H)$, and the presence or absence of an infinite cluster in the lattice when $p=\frac{1}{2}$. This may have some relevance to the confusion regarding the lower critical dimension d_l mentioned above.

An interesting difference from the standard Ising model arises in the nature of overturned clusters as the field is increased. For the standard ferromagnetic model, in a small field single spins can overturn, and as the field increases larger and larger clusters are able to overturn. For the random field model the largest clusters can overturn for the smallest fields, and the highest field is required to overturn single spins. When we consider the dynamics of the model it will be difficult to find processes for which large numbers of spins can overturn simultaneously, and the indications are clear for significant metastabilities.

Some of the above considerations apply to the spin-glass problem for which the associated clusters involve a bond percolation process. We hope to discuss this in a subsequent communication.

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Note added in proof. The situation as depicted in § 2 represents an oversimplification. Each percolation cluster need not overturn as a whole. It will do so only if it is more compact than any of its sub-clusters, otherwise it may split. For example in a cluster with a compact core and tentacles, the compact core will overturn at a lower field than the tentacles. However, this does not affect the basic pattern of behaviour described.

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